Published by IOP Publishing for SISSA
Received: *November 12, 2008*
Accepted: *December 8, 2008*
Published: *December 16, 2008*

Chiral formulation for hyperKähler sigma-models on cotangent bundles of symmetric spaces

Sergei M. Kuzenko and Joseph Novak

School of Physics M013, The University of Western Australia,
35 Stirling Highway, Crawley W.A. 6009, Australia
E-mail: kuzenko@cyllene.uwa.edu.au, novakj02@tartarus.uwa.edu.au

Abstract: Starting with the projective-superspace off-shell formulation for four-dimensional $\mathcal{N} = 2$ supersymmetric sigma-models on cotangent bundles of *arbitrary* Hermitian symmetric spaces, their on-shell description in terms of $\mathcal{N} = 1$ chiral superfields is developed. In particular, we derive a universal representation for the hyperkähler potential in terms of the curvature of the symmetric base space. Within the tangent-bundle formulation for such sigma-models, completed recently in arXiv:0709.2633 and realized in terms of $\mathcal{N} = 1$ chiral and complex linear superfields, we give a new universal formula for the superspace Lagrangian. A closed form expression is also derived for the Kähler potential of an arbitrary Hermitian symmetric space in Kähler normal coordinates.

Keywords: Extended Supersymmetry, Superspaces.

JHEP12(2008)072

Contents

1. Introduction

Ten years ago, it was noticed [1, 2], using the projective-superspace techniques [3], that the general four-dimensional $\mathcal{N} = 1$ supersymmetric nonlinear sigma-model [4]

$$S[\Phi, \bar{\Phi}] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K(\Phi^I, \bar{\Phi}^{\bar{J}})\,, \tag{1.1}$$

with K the Kähler potential of a Kähler manifold $\mathcal{M}$, admits an off-shell $\mathcal{N} = 2$ extension formulated in $\mathcal{N} = 1$ superspace as follows:

$$S[\Upsilon, \breve{\Upsilon}] = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K\big(\Upsilon^I(\zeta), \breve{\Upsilon}^{\bar{J}}(\zeta)\big)\ . \tag{1.2}$$

Here $\zeta \in \mathbb{C} \setminus 0$ is an auxiliary complex variable, and the dynamical variables $\Upsilon(\zeta)$ and $\breve{\Upsilon}(\zeta)$ comprise an infinite set of ordinary $\mathcal{N} = 1$ superfields:

$$\Upsilon(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n = \Phi + \Sigma\,\zeta + O(\zeta^2)\,, \qquad \breve{\Upsilon}(\zeta) = \sum_{n=0}^{\infty} \bar{\Upsilon}_n (-\zeta)^{-n}\,, \tag{1.3}$$

with Φ chiral, Σ complex linear,

$$\bar{D}_{\dot\alpha} \Phi = 0\,, \qquad \bar{D}^2 \Sigma = 0\,, \tag{1.4}$$

and the remaining component, $\Upsilon_2, \Upsilon_3, \ldots$, being unconstrained complex superfields.[1] The latter enter the action without derivatives, and therefore they form an infinite set of auxiliary superfields. As pointed out in [1], the $\mathcal{N} = 2$ supersymetric sigma-model (1.2) inherits all the geometric features of its $\mathcal{N} = 1$ predecessor (1.1), that is properly realised Kähler symmetry and invariance under holomorphic reparametrizations of the Kähler manifold.

[1] In the terminology of [5], the superfields $\Upsilon(\zeta)$ and $\breve{\Upsilon}(\zeta)$ realize a polar hypermultiplet. The most general $\mathcal{N} = 2$ supersymmetric sigma-model couplings of polar hypermultiplets [3] are obtained from (1.2) by allowing K to depend explicitly on ζ, $K\big(\Upsilon, \breve{\Upsilon}\big) \to K\big(\Upsilon, \breve{\Upsilon}, \zeta\big)$. A geometric interpretation of such generalized couplings has recently been discussed in [6].

The latter property implies that the variables (Φ^I, Σ^J) parametrize the tangent bundle $T\mathcal{M}$ of the Kähler manifold $\mathcal{M}$ [1].

The auxiliary superfields $\Upsilon_2, \Upsilon_3, \dots$ can in principle be integrated out, at least in perturbation theory, and then the action (1.2) turns into [2]

$$S_{\rm tb}[\Phi, \Sigma] = \int d^4x\, d^4\theta \left\{ K\big(\Phi, \bar{\Phi}\big) + \mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big) \right\}, \tag{1.5}$$

where the second term in the Lagrangan looks like

$$\mathcal{L} = \sum_{n=1}^{\infty} \mathcal{L}^{(n)}, \qquad \mathcal{L}^{(n)} = \mathcal{L}_{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}\big(\Phi, \bar{\Phi}\big) \Sigma^{I_1} \dots \Sigma^{I_n} \bar{\Sigma}^{\bar{J}_1} \dots \bar{\Sigma}^{\bar{J}_n}. \tag{1.6}$$

Here $\mathcal{L}_{I\bar{J}} = -g_{I\bar{J}}\big(\Phi, \bar{\Phi}\big)$, while the tensors $\mathcal{L}_{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}$ for $n > 1$ are functions of the Riemann curvature $R_{I\bar{J}K\bar{L}}\big(\Phi, \bar{\Phi}\big)$ and its covariant derivatives.

The theory with action (1.5) possesses a dual formulation. It can be obtained by considering the first-order action

$$S = \int d^4x\, d^4\theta \left\{ K\big(\Phi, \bar{\Phi}\big) + \mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big) + \Psi_I\, \Sigma^I + \bar{\Psi}_{\bar{I}} \bar{\Sigma}^{\bar{I}} \right\}, \tag{1.7}$$

where the tangent vector Σ^I is now complex unconstrained, while the one-form Ψ_I is chiral, $\bar{D}_{\dot{\alpha}} \Psi_I = 0$. Integrating out Σ's and their conjugates gives

$$S_{\rm ctb}[\Phi, \Psi] = \int d^4x\, d^4\theta \left\{ K\big(\Phi, \bar{\Phi}\big) + \mathcal{H}\big(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}\big) \right\}, \tag{1.8}$$

where the second term in the Lagrangian is

$$\mathcal{H} = \sum_{n=1}^{\infty} \mathcal{H}^{(n)}, \quad \mathcal{H}^{(n)} = \mathcal{H}^{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}\big(\Phi, \bar{\Phi}\big) \Psi_{I_1} \dots \Psi_{I_n} \bar{\Psi}_{\bar{J}_1} \dots \bar{\Psi}_{\bar{J}_n}, \tag{1.9}$$

with $\mathcal{H}^{I\bar{J}} = g^{I\bar{J}}\big(\Phi, \bar{\Phi}\big)$. The variables (Φ^I, Ψ_J) parametrize the cotangent bundle $T^*\mathcal{M}$ of the Kähler manifold $\mathcal{M}$ [2]. Since the theory with action (1.8) is $\mathcal{N} = 2$ supersymmetric and realized in terms of chiral superfields, the Lagrangian in (1.8) constitutes the hyperkähler potential for (in general, an open domain of the zero section of) $T^*\mathcal{M}$, in accordance with [7]. If $\mathcal{M}$ is a compact Hermitian symmetric space, then the hyperkähler structure turns out to be globally defined on $T^*\mathcal{M}$.

The problem of explicit computation of $\mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big)$ and $\mathcal{H}\big(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}\big)$ from the off-shell sigma-model (1.2) was addressed in a series of papers [2, 8−11] for the case when $\mathcal{M}$ is a Hermitian symmetric space. The method[2] used in [2, 8−10] essentially exploited the property of such a manifold $\mathcal{M}$ to be a homogeneous space with respect to an appropriate Lie group of holomorphic isometries. Being perfectly viable, such a setting has

[2]The method was introduced in [2] and illustrated on the example of $\mathcal{M} = \mathbb{C}P^1$. The case of $\mathbb{C}P^n$ was worked out in [8, 9]. The classical compact symmetric spaces $\mathrm{U}(n+m)/\mathrm{U}(n) \times \mathrm{U}(m)$, $\mathrm{SO}(2n)/\mathrm{U}(n)$, $SP(n)/\mathrm{U}(n)$ and $\mathrm{SO}(n+2)/\mathrm{SO}(n) \times \mathrm{SO}(2)$, as well as their non-compact versions, were worked out in [10]. The tangent-bundle formulation for $\mathrm{SO}(n+2)/\mathrm{SO}(n) \times \mathrm{SO}(2)$ was given for the first time in [9].

JHEP12(2008)072

a minor disadvantage in the sense that it requires a separate consideration for different Hermitian symmetric spaces, on case by case basis. In particular, this method becomes somewhat cumbersome in the case of exceptional symmetric spaces including the compact ones $E_6/\mathrm{SO}(10) \times \mathrm{U}(1)$ and $E_7/E_6 \times \mathrm{U}(1)$. To address the latter spaces, the conceptual set-up was changed in ref. [11], which built on the property of any Hermitian symmetric spaces that its curvature tensor is covariantly constant,

$$\nabla_L R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = \bar{\nabla}_{\bar{L}} R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = 0 \ . \tag{1.10}$$

In conjunction with supersymmetry considerations, this idea allowed the authors of [11] to derive the following closed form expression for the tangent-bundle Lagrangian:

$$\mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big) = -g_{I\bar{J}} \bar{\Sigma}^{\bar{J}} \, \frac{\mathrm{e}^{\mathcal{R}_{\Sigma,\bar{\Sigma}}} - 1}{\mathcal{R}_{\Sigma,\bar{\Sigma}}} \, \Sigma^I \, , \qquad \mathcal{R}_{\Sigma,\bar{\Sigma}} = -\frac{1}{2} \Sigma^K \bar{\Sigma}^{\bar{L}} \, R_{K\bar{L}I}{}^J \, \Sigma^I \frac{\partial}{\partial \Sigma^J} \ . \tag{1.11}$$

Using this representation, the case of $E_6/\mathrm{SO}(10)\times\mathrm{U}(1)$ was worked out in [11] for the first time.[3] However, no universal closed form expression for $\mathcal{H}\big(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}\big)$ was found in [11]. One of the aims of the present work is to fill this gap.

This paper is organized as follows. In section 2, we derive an alternative closed form expression for $\mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big)$ which differs from (1.11). The specific feature of this new representation is that the curvature tensor appears in it as a matrix, unlike the differential operator in eq. (1.11). In section 3, we derive $\mathcal{H}\big(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}\big)$ in a closed form. Finally, the appendix is devoted to deriving a closed form expression for the Kähler potential of an arbitrary Hermitian symmetric space in so-called Kähler normal coordinates (or Bochner's canonical coordinates) [13, 14]. In the main body of the paper, the Kähler manifold $\mathcal{M}$ is only assumed to obey eq. (1.10).

A few words are in order regarding the content of the appendix. Recently, an intimate connection was pointed out in ref. [12] between the tangent-bundle Lagrangian $\mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big)$ in (1.5) and the Kähler potential $K(\phi, \bar{\phi})$ for $\mathcal{M}$ given in Kähler normal coordinates ϕ with origin at Φ. In the symmetric case, eq. (1.10), this correspondence is as follows:

$$\mathcal{L}\big(\Sigma, \bar{\Sigma}\big) = K(\phi \to -\Sigma \, , \, \bar{\phi} \to \bar{\Sigma}) \ . \tag{1.12}$$

The derivation of eq. (1.11) in [11], or the equivalent representation (2.6) below, are based on supersymmetry consideration. Due to (1.12), there should exist a purely geometric way of deriving analogues of the representations (1.11) and (2.6) for $K(\phi, \bar{\phi})$. It is presented in the appendix.

2. Tangent-bundle formulation

The Lagrangian $\mathcal{L}\big(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\big)$ obeys the first-order linear differential equation [11]

$$\frac{1}{2} \Sigma^K \Sigma^L \, R_{K\bar{J}L}{}^I \, \mathcal{L}_I + \mathcal{L}_{\bar{J}} + g_{I\bar{J}} \, \Sigma^I = 0 \, , \qquad \mathcal{L}_I := \frac{\partial \mathcal{L}}{\partial \Sigma^I} \tag{2.1}$$

[3] The tangent-bundle formulation for $E_7/E_6 \times \mathrm{U}(1)$ was sketched in [12].

JHEP12(2008)072

and its conjugate. As demonstrated in [11], this equation expresses the fact that the theory (1.5) is $\mathcal{N}=2$ supersymmetric. It can be shown that this equation is identically satisfied by the function (1.11). A different representation for this solution is provided below.

It proves robust to rewrite (2.1) in a matrix form. For this purpose, we introduce the following matrices:

$$\boldsymbol{R}_{\Sigma,\bar\Sigma} := \begin{pmatrix} 0 & (R_\Sigma)^I{}_{\bar J} \\ (R_{\bar\Sigma})^{\bar I}{}_J & 0 \end{pmatrix}, \qquad (R_\Sigma)^I{}_{\bar J} := \frac{1}{2} R_K{}^I{}_{L\bar J}\,\Sigma^K\Sigma^L, \quad (R_{\bar\Sigma})^{\bar I}{}_J := \overline{(R_\Sigma)^I{}_{\bar J}} \tag{2.2}$$

and

$$\boldsymbol{g} := \begin{pmatrix} 0 & g_{I\bar J} \\ g_{\bar I J} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \hat g \\ \check g & 0 \end{pmatrix}. \tag{2.3}$$

Then eq. (2.1) is equivalent to

$$\begin{pmatrix} \mathcal{L}_I \\ \mathcal{L}_{\bar I} \end{pmatrix} = -\boldsymbol{g}\Big(\mathbb{1} + \boldsymbol{R}_{\Sigma,\bar\Sigma}\Big)^{-1} \begin{pmatrix} \Sigma^I \\ \Sigma^{\bar I} \end{pmatrix}. \tag{2.4}$$

This relation actually allows one to determine $\mathcal{L}$ by taking into account the identities

$$\Sigma^I \mathcal{L}_I = \bar\Sigma^{\bar I}\mathcal{L}_{\bar I} = \sum_{n=1}^{\infty} n\,\mathcal{L}^{(n)}. \tag{2.5}$$

One then obtains

$$\mathcal{L}\big(\Phi,\bar\Phi,\Sigma,\bar\Sigma\big) = -\frac{1}{2}\boldsymbol{\Sigma}^{\rm T}\boldsymbol{g}\,\frac{\ln\big(\mathbb{1}+\boldsymbol{R}_{\Sigma,\bar\Sigma}\big)}{\boldsymbol{R}_{\Sigma,\bar\Sigma}}\,\boldsymbol{\Sigma}, \qquad \boldsymbol{\Sigma} := \begin{pmatrix} \Sigma^I \\ \bar\Sigma^{\bar I} \end{pmatrix}. \tag{2.6}$$

It also follows from (2.4) that the following composites

$$\begin{aligned} F^{(2k+2)} &:= \Sigma^{\rm T}\hat g (R_{\bar\Sigma}R_\Sigma)^k R_{\bar\Sigma}\Sigma = \Sigma^\dagger \check g (R_\Sigma R_{\bar\Sigma})^k R_\Sigma \bar\Sigma, \qquad && k=0,1,2,\dots \\ F^{(2k+1)} &:= \Sigma^{\rm T}\hat g (R_{\bar\Sigma}R_\Sigma)^k \bar\Sigma = \Sigma^\dagger \check g (R_\Sigma R_{\bar\Sigma})^k \Sigma, && k=0,1,2,\dots \end{aligned} \tag{2.7}$$

which appear in the Taylor expansion of $\mathcal{L}\big(\Phi,\bar\Phi,\Sigma,\bar\Sigma\big)$, have the properties

$$F_I^{(2k+2)} = (2k+2)\hat g (R_{\bar\Sigma}R_\Sigma)^k R_{\bar\Sigma}\Sigma, \qquad F_I^{(2k+1)} = (2k+1)\hat g (R_{\bar\Sigma}R_\Sigma)^k \bar\Sigma. \tag{2.8}$$

Eq. (2.6) constitutes our new closed form expression for $\mathcal{L}$, compare with (1.11).

In [11], it was conjectured that $\mathcal{L}\big(\Phi,\bar\Phi,\Sigma,\bar\Sigma\big)$ can be represented in the form

$$\mathcal{L}\big(\Phi,\bar\Phi,\Sigma,\bar\Sigma\big) = -\Sigma^\dagger \hat g\,\frac{\ln\big(\mathbb{1}+\mathbb{R}_{\Sigma,\bar\Sigma}\big)}{\mathbb{R}_{\Sigma,\bar\Sigma}}\,\Sigma, \qquad (\mathbb{R}_{\Sigma,\bar\Sigma})^I{}_J := \frac{1}{2} R_J{}^I{}_{K\bar L}\Sigma^K\bar\Sigma^{\bar L} \tag{2.9}$$

which differs from (2.6). The validity of this representation was checked in [11] for the followings two choices of $\mathcal{M}$: (i) $\mathbb{C}P^n$; and (ii) $\mathrm{SO}(n+2)/\mathrm{SO}(n)\times\mathrm{SO}(2)$. Unlike the representation (2.6), we still do not have a proof that (2.9) holds in general (however, see comments at the end of the next section).

Using the correspondence (1.12) and Kähler normal coordinate considerations (see the appendix), one can derive an alternative second-order differential equation enjoyed by $\mathcal{L}$:

$$\mathcal{L}_{IJ} = \frac{1}{2} R_I{}^K{}_J{}^L\,\mathcal{L}_K\mathcal{L}_L. \tag{2.10}$$

3. Cotangent-bundle formulation

The "Hamiltonian" $\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ obeys the nonlinear differential equation [11]

$$\mathcal{H}^I \, g_{I\bar{J}} - \frac{1}{2} \mathcal{H}^K \mathcal{H}^L \, R_{K\bar{J}L}{}^I \, \Psi_I = \bar{\Psi}_{\bar{J}} \,, \qquad \mathcal{H}^I = \frac{\partial \mathcal{H}}{\partial \Psi_I} \ . \tag{3.1}$$

This equation immediately follows from (2.1) if one makes use of the standard properties of the Legendre transformation. Alternatively, eq. (3.1) is equivalent to the condition that the cotangent-bundle action (1.8) is $\mathcal{N}=2$ supersymmetric [11]. The hidden SUSY transformation, which is not manifest in the $\mathcal{N}=1$ superspace formulation, is [11]:

$$\begin{aligned} \delta\Phi^I &= \frac{1}{2}\bar{D}^2\big\{\overline{\varepsilon\theta}\,\mathcal{H}^I\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\big\}\,, \\ \delta\Psi_I &= -\frac{1}{2}\bar{D}^2\Big\{\overline{\varepsilon\theta}\,K_I\big(\Phi,\bar{\Phi}\big)\Big\} + \frac{1}{2}\bar{D}^2\Big\{\overline{\varepsilon\theta}\,\Gamma^K_{IJ}\big(\Phi,\bar{\Phi}\big)\,\mathcal{H}^J\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\Big\}\,\Psi_K\,, \end{aligned} \tag{3.2}$$

with Γ^K_{IJ} the Christoffel symbols for the Kähler metric. The nonlinearity of (3.1) makes it more difficult to solve than (2.1). Below we provide the solution to eq. (3.1).

Equation (3.1) implies

$$\Psi_I \mathcal{H}^I - \mathcal{H}^K \mathcal{H}^L \, (R_\Psi)_{KL} = g^{I\bar{J}} \Psi_I \bar{\Psi}_{\bar{J}} \,, \qquad (R_\Psi)_{KL} := \frac{1}{2} R_K{}^I{}_L{}^J \, \Psi_I \Psi_J \ . \tag{3.3}$$

Due to the identities

$$\Psi_I \mathcal{H}^I = \bar{\Psi}_{\bar{I}} \mathcal{H}^{\bar{I}} = \sum_{n=1}^{\infty} n \, \mathcal{H}^{(n)} \,, \tag{3.4}$$

the latter equation is equivalent to the following infinite system of equations

$$\mathcal{H}^{(1)} = g^{I\bar{J}} \Psi_I \bar{\Psi}_{\bar{J}} \,, \qquad n\mathcal{H}^{(n)} - \sum_{p=1}^{n-1} \mathcal{H}^{(p)\,K} (R_\Psi)_{KL} \mathcal{H}^{(n-p)\,L} = 0 \,, \quad n \geq 2 \ . \tag{3.5}$$

It is clear that the contributions $\mathcal{H}^{(2)}, \mathcal{H}^{(3)}, \ldots,$ can be uniquely determined, order by order in perturbation theory, using the equations derived.

To solve (3.5), it is useful to introduce a matrix associated with the Riemann tensor

$$\boldsymbol{R}_{\Psi,\bar{\Psi}} := \begin{pmatrix} 0 & (R_\Psi)_I{}^{\bar{J}} \\ (R_{\bar{\Psi}})_{\bar{I}}{}^J & 0 \end{pmatrix} \,, \qquad (R_\Psi)_I{}^{\bar{J}} = (R_\Psi)_{IK} \, g^{K\bar{J}} \,, \tag{3.6}$$

as well as a family of building blocks

$$\begin{aligned} G^{(2k+2)} &:= \Psi^{\mathrm{T}} \hat{g}^{-1} (R_{\bar{\Psi}} R_\Psi)^k R_{\bar{\Psi}} \Psi = \Psi^\dagger \check{g}^{-1} (R_\Psi R_{\bar{\Psi}})^k R_\Psi \bar{\Psi} \,, \qquad k = 0,1,2,\ldots \\ G^{(2k+1)} &:= \Psi^{\mathrm{T}} \hat{g}^{-1} (R_{\bar{\Psi}} R_\Psi)^k \bar{\Psi} = \Psi^\dagger \check{g}^{-1} (R_\Psi R_{\bar{\Psi}})^k \Psi \,, \qquad k = 0,1,2,\ldots \ . \end{aligned} \tag{3.7}$$

Their partial derivatives can be read off from (2.8)

$$\begin{aligned} G^{(2k+2)\,I} &:= (2k+2) \hat{g}^{-1} (R_{\bar{\Psi}} R_\Psi)^k R_{\bar{\Psi}} \Psi = (2k+2) \Psi^{\mathrm{T}} \hat{g}^{-1} (R_{\bar{\Psi}} R_\Psi)^k R_{\bar{\Psi}} \,, \\ G^{(2k+1)\,I} &:= (2k+1) \hat{g}^{-1} (R_{\bar{\Psi}} R_\Psi)^k \bar{\Psi} = (2k+1) \Psi^\dagger \hat{g}^{-1} (R_\Psi R_{\bar{\Psi}})^k \ . \end{aligned} \tag{3.8}$$

Now, if one introduces an ansatz

$$\mathcal{H}^{(n)} = c_n \, G^{(n)} \,, \qquad n \geq 2 \tag{3.9}$$

with c_n numerical coefficients, the equations (3.5) turn into the following system of quadratic algebraic equations:

$$n \, c_n - \sum_{p=1}^{n-1} p(n-p) \, c_p c_{n-p} = 0 \,, \qquad c_1 = 1 \ . \tag{3.10}$$

The algebraic equations (3.10) are universal and independent of the symmetric space $\mathcal{M}$ chosen. Therefore, their solution can be deduced by considering any useful choice of $\mathcal{M}$, for which $\mathcal{H}$ is known, say the projective space $\mathbb{C}P^n$ first considered by Calabi [15]. This observation immediately leads to the solution

$$\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \frac{1}{2} \boldsymbol{\Psi}^{\mathrm{T}} \boldsymbol{g}^{-1} \mathcal{F}\Big(- \boldsymbol{R}_{\Psi, \bar{\Psi}} \Big) \, \boldsymbol{\Psi} \,, \qquad \boldsymbol{\Psi} := \begin{pmatrix} \Psi_I \\ \bar{\Psi}_{\bar{I}} \end{pmatrix} \,, \tag{3.11}$$

where

$$\mathcal{F}(x) = \frac{1}{x} \left\{ \sqrt{1+4x} - 1 - \ln \frac{1+\sqrt{1+4x}}{2} \right\} , \qquad \mathcal{F}(0) = 1 \ . \tag{3.12}$$

Eq. (3.11) is the main result of this work.

To write down the supersymmetry transformation (3.2) explicitly, we need to compute $\mathcal{H}^I$ and its conjugate. Direct calculations give

$$\begin{pmatrix} \mathcal{H}^I \\ \mathcal{H}^{\bar{I}} \end{pmatrix} = -\frac{1}{2} \boldsymbol{g}^{-1} \frac{\sqrt{\mathbb{1} - 4 \boldsymbol{R}_{\Psi, \bar{\Psi}}} - \mathbb{1}}{\boldsymbol{R}_{\Psi, \bar{\Psi}}} \begin{pmatrix} \Psi_I \\ \bar{\Psi}_{\bar{I}} \end{pmatrix} \ . \tag{3.13}$$

Our derivation of the hyperkähler potential for $T^*\mathcal{M}$,

$$K(\Phi, \bar{\Phi}) + \frac{1}{2} \boldsymbol{\Psi}^{\mathrm{T}} \boldsymbol{g}^{-1} \mathcal{F}\Big(- \boldsymbol{R}_{\Psi, \bar{\Psi}} \Big) \, \boldsymbol{\Psi} \,, \tag{3.14}$$

was based on the considerations of extended supersymmetry. In the mathematical literature, there exists a different representation for $\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ [16]:

$$\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \Psi^{\dagger} \check{g}^{-1} \mathcal{F}\Big(- \mathbb{R}_{\Psi, \bar{\Psi}} \Big) \, \Psi \,, \qquad (\mathbb{R}_{\Psi, \bar{\Psi}})_I{}^J := \frac{1}{2} R_I{}^{J \bar{K} L} \Psi_L \bar{\Psi}_{\bar{K}} \ . \tag{3.15}$$

This unified formula was derived by Biquard and Gauduchon by using purely algebraic means involving the root theory for Hermitian symmetric spaces. It should be pointed out that the operator $\mathbb{R}_{\Psi, \bar{\Psi}}$ above is related to $\mathbb{R}_{\Sigma, \bar{\Sigma}}$ appearing in (2.9). It is worth expecting that similar algebraic arguments can be used to prove the validity of (2.9) for any Hermitian symmetric space.

The $\mathcal{N} = 2$ supersymmetric model on $T^*\mathcal{M}$ constructed above can be generalized to include a superpotential consistent with $\mathcal{N} = 2$ supersymmetry. In accordance with the analysis in [17] (see also [18]), the superpotential is

$$\mathrm{e}^{\mathrm{i}\sigma} \int \mathrm{d}^2\theta \, \Psi_I \, X^I(\Phi) \ + \ \mathrm{c.c.} \,, \tag{3.16}$$

where $\mathrm{e}^{\mathrm{i}\sigma}$ is a constant phase factor, and $X^I(\Phi)$ a holomorphic Killing vector of the base Kähler manifold $\mathcal{M}$. Similar results hold in five space-time dimensions [17, 18].

JHEP12(2008)072

Acknowledgments

This work is supported in part by the Australian Research Council.

A. Kähler normal coordinates

Let us recall the important notion of a *canonical* coordinate system for a Kähler manifold, that was introduced by Bochner in 1947 [13] and later used by Calabi in the 1950s [14].[4] In a neighborhood of any point p of the Kähler manifold $\mathcal{M}$, holomorphic reparametrizations and Kähler transformations can be used to choose a coordinate system, with origin at $p \in \mathcal{M}$, in which the Kähler potential takes the form:

$$\begin{aligned} K(\phi,\bar\phi) &= g_{I\bar J}|\,\phi^I\bar\phi^{\bar J} + \sum_{m,n\geq 2}^{\infty} K^{(m,n)}(\phi,\bar\phi)\,, \\ K^{(m,n)}(\phi,\bar\phi) &:= \frac{1}{m!n!}\,K_{I_1\cdots I_m\bar J_1\cdots\bar J_n}|\,\phi^{I_1}\dots\phi^{I_m}\bar\phi^{\bar J_1}\dots\bar\phi^{\bar J_n}\,. \end{aligned} \tag{A.1}$$

In such a coordinate system, there still remains the freedom to perform linear holomorphic reparametrizations which can be used to set the metric at the origin to be $g_{I\bar J} = \delta_{I\bar J}$. The Taylor coefficients in (A.1), $K_{I_1\cdots I_m\bar J_1\cdots\bar J_n}|$, turn out to be tensor functions of the Kähler metric, the Riemann curvature $R_{I\bar J K\bar L}$ and its covariant derivatives, all of which are evaluated at the origin. In the physics literature, Bochner's canonical coordinates are often called "Kähler normal coordinates" [22]. We follow this terminology. Kähler normal coordinates are very useful for various considerations, in particular in the context of the so-called Bergman kernel [23].

In the case of symmetric spaces,

$$\nabla_L R_{I_1\bar J_1 I_2\bar J_2} = \bar\nabla_{\bar L} R_{I_1\bar J_1 I_2\bar J_2} = 0 \qquad \Longrightarrow \qquad K^{(m,n)} = 0\,, \quad m\neq n\,. \tag{A.2}$$

The condition of covariant constancy can be rewritten as

$$\bar\nabla_{\bar L} R_{I_1}{}^{J_1}{}_{I_2}{}^{J_2} = \bar\partial_{\bar L} R_{I_1}{}^{J_1}{}_{I_2}{}^{J_2} = 0\,, \tag{A.3}$$

and therefore $R_{I_1}{}^{J_1}{}_{I_2}{}^{J_2}$ is $\bar\phi$-independent. Since

$$R_{I_1\bar J_1 I_2\bar J_2} = K_{I_1 I_2\bar J_1\bar J_2} - g_{M\bar N}\Gamma^M_{I_1 I_2}\bar\Gamma^{\bar N}_{\bar J_1\bar J_2} = K_{I_1 I_2\bar J_1\bar J_2} - g^{\bar M N}K_{I_1 I_2\bar M}K_{N\bar J_1\bar J_2}\,, \tag{A.4}$$

and terms in the Taylor series for the expression on the right involve equal numbers of ϕ and $\bar\phi$, we conclude[5]

$$R_{I_1}{}^{J_1}{}_{I_2}{}^{J_2} = R_{I_1}{}^{J_1}{}_{I_2}{}^{J_2}| = \text{const}\,. \tag{A.5}$$

Then we can write

$$\Gamma^M_{I_1 I_2,\bar J} = R_{I_1}{}^M{}_{I_2\bar J} = R_{I_1}{}^M{}_{I_2}{}^N g_{N\bar J} = R_{I_1}{}^M{}_{I_2}{}^N K_{N\bar J}\,, \tag{A.6}$$

[4] This coordinate system was re-discovered by supersymmetry practitioners in the 1980s under the name *normal gauge* [19 – 21].

[5] For the Hermitian symmetric space $\mathcal{M} = G/H$, the constant tensor $R_{I_1}{}^{J_1}{}_{I_2}{}^{J_2}$ can be related to the structure constants of G.

JHEP12(2008)072

and hence

$$\Gamma^M_{I_1 I_2} = R_{I_1}{}^M{}_{I_2}{}^N K_N \ . \tag{A.7}$$

Contracting both sides of this equation with the metric, $g_{M\bar{Q}}$, one can arrive at the equation

$$K_{I_1 I_2} = \frac{1}{2} R_{I_1}{}^M{}_{I_2}{}^N K_M K_N \ . \tag{A.8}$$

Equation (A.8) is highly important, since it makes it possible to uniquely restore $K(\phi, \bar{\phi})$ provided its functional form, eqs. (A.1) and (A.2), is taken into account. In particular, using eq. (A.8) allows one to deduce the following alternative equation:

$$g_{I\bar{J}}|\, \phi^I + \frac{1}{2}\phi^K \phi^L R_{K\bar{J}L}{}^I|\, K_I = K_{\bar{J}} \ . \tag{A.9}$$

For the Kähler potential, one obtains the following closed form expression:

$$K\big(\phi, \bar{\phi}\big) = -\frac{1}{2}\boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{g}| \, \frac{\ln\big(\mathbb{1} - \boldsymbol{R}_{\phi,\bar{\phi}}\big)}{\boldsymbol{R}_{\phi,\bar{\phi}}} \, \boldsymbol{\phi} \,, \qquad \boldsymbol{\phi} := \begin{pmatrix} \phi^I \\ \bar{\phi}^{\bar{I}} \end{pmatrix} \ . \tag{A.10}$$

Here $\boldsymbol{R}_{\phi,\bar{\phi}}$ is obtained from (2.2) by replacing $\Sigma \to \phi$ and $R_K{}^I{}_{L\bar{J}} \to R_K{}^I{}_L{}^J g_{J\bar{J}}|$.

We should emphasize that our derivation above only relied on eq. (A.2).

References

[1] S.M. Kuzenko, *Projective superspace as a double-punctured harmonic superspace*, *Int. J. Mod. Phys.* **A 14** (1999) 1737 [hep-th/9806147].

[2] S.J. Gates Jr. and S.M. Kuzenko, *The CNM-hypermultiplet nexus*, *Nucl. Phys.* **B 543** (1999) 122 [hep-th/9810137].

[3] U. Lindström and M. Roček, *New hyperKähler metrics and new supermultiplets*, *Commun. Math. Phys.* **115** (1988) 21; *$N = 2$ super Yang-Mills theory in projective superspace*, *Commun. Math. Phys.* **128** (1990) 191.

[4] B. Zumino, *Supersymmetry and Kähler Manifolds*, *Phys. Lett.* **B 87** (1979) 203.

[5] F. Gonzalez-Rey, M. Roček, S. Wiles, U. Lindström and R. von Unge, *Feynman rules in $N = 2$ projective superspace. I: massless hypermultiplets*, *Nucl. Phys.* **B 516** (1998) 426 [hep-th/9710250].

[6] U. Lindström and M. Roček, *Properties of hyperKähler manifolds and their twistor spaces*, arXiv:0807.1366.

[7] L. Álvarez-Gaumé and D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric σ-model*, *Commun. Math. Phys.* **80** (1981) 443.

[8] S.J. Gates Jr. and S.M. Kuzenko, *4D $N = 2$ supersymmetric off-shell σ-models on the cotangent bundles of Kähler manifolds*, *Fortschr. Phys.* **48** (2000) 115 [hep-th/9903013].

[9] M. Arai and M. Nitta, *Hyper-Kähler σ-models on (co)tangent bundles with* SO(n) *isometry*, *Nucl. Phys.* **B 745** (2006) 208 [hep-th/0602277].

JHEP12(2008)072

[10] M. Arai, S.M. Kuzenko and U. Lindström, *HyperKähler σ-models on cotangent bundles of Hermitian symmetric spaces using projective superspace*, *JHEP* **02** (2007) 100 [hep-th/0612174].

[11] M. Arai, S.M. Kuzenko and U. Lindström, *Polar supermultiplets, Hermitian symmetric spaces and hyperKähler metrics*, *JHEP* **12** (2007) 008 [arXiv:0709.2633].

[12] S.M. Kuzenko, *On superconformal projective hypermultiplets*, *JHEP* **12** (2007) 010 [arXiv:0710.1479].

[13] S. Bochner, *Curvature in Hermitian metric*, *Bull. Amer. Math. Soc.* **53** (1947) 179.

[14] E. Calabi, *Isometric imbedding of complex manifolds*, *Ann. of Math.* **58** (1953) 1; *On compact, locally symmetric Kähler manifolds*, *Ann. of Math.* **71** (1960) 472.

[15] E. Calabi, *Métriques kählériennes et fibrés holomorphes*, *Ann. Sci. École Norm. Sup.* **12** (1979) 269.

[16] O. Biquard and P. Gauduchon, *Hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces*, in *Geometry and Physics*, J. Andersen et al., Marcel Dekker, U.S.A. (1997), pag. 287.

[17] S.M. Kuzenko, *On superpotentials for nonlinear σ-models with eight supercharges*, *Phys. Lett.* **B 638** (2006) 288 [hep-th/0602050].

[18] J. Bagger and C. Xiong, *$N = 2$ nonlinear σ-models in $N = 1$ superspace: four and five dimensions*, hep-th/0601165.

[19] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [hep-th/0108200].

[20] L. Álvarez-Gaumé and P.H. Ginsparg, *Finiteness of Ricci flat supersymmetric nonlinear σ-models*, *Commun. Math. Phys.* **102** (1985) 311.

[21] C.M. Hull, A. Karlhede, U. Lindström and M. Roček, *Nonlinear σ-models and their gauging in and out of superspace*, *Nucl. Phys.* **B 266** (1986) 1.

[22] K. Higashijima, E. Itou and M. Nitta, *Normal coordinates in Kähler manifolds and the background field method*, *Prog. Theor. Phys.* **108** (2002) 185 [hep-th/0203081].

[23] M.R. Douglas and S. Klevtsov, *Bergman kernel from path integral*, arXiv:0808.2451.

JHEP12(2008)072